Irrationality via the Hypergeometric method

Michael A. Bennett

Department of Mathematics, University of British Columbia, Vancouver, BC Canada V6T 1Z2
bennett@math.ubc.ca

Abstract. In this paper, we describe how the hypergeometric method of Thue and Siegel may be applied to questions of irrationality. As a consequence of our approach, we provide a somewhat simple proof of a classical theorem of Ljunggren to the effect that the Diophantine equation $x^2 - 2y^4 = -1$ has only the solutions $(x,y) = (1,1)$ and $(x,y) = (239,13)$ in positive integers.
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INTRODUCTION

Suppose that we are given a real number θ that we wish to prove to be irrational. One way to do this is to find a sequence of distinct rational approximations p_n/q_n to θ (here, p_n and q_n are integers) with the property that there exist positive real numbers α, β, a and b with $\alpha, \beta > 1$,

$$|q_n| < a \cdot \alpha^n, \text{ and } |q_n \theta - p_n| < b \cdot \beta^{-n},$$

for all positive integers n. If we can construct such a sequence, we in fact get rather more. Namely, we obtain the inequality

$$\left| \theta - \frac{p}{q} \right| > \left(2 a \alpha (2 b \beta)^{\lambda} \right)^{-1} |q|^{-1-\lambda} \quad \text{for} \quad \lambda = \frac{\log \alpha}{\log \beta}, \tag{1}$$

valid for *all* integers p and $q \neq 0$ (at least provided $|q| > 1/2b$). To see this, note that

$$\left| \theta - \frac{p}{q} \right| \geq \left| \frac{p_n}{q_n} - \frac{p}{q} \right| - \left| \theta - \frac{p_n}{q_n} \right|,$$

and hence if $p/q \neq p_n/q_n$, we have

$$\left| \theta - \frac{p}{q} \right| > \frac{1}{|q_n|} \left(\frac{1}{|q|} - \frac{b}{\beta^n} \right).$$

Now we choose n minimal such that $\beta^n \geq 2b\,|q|$ (since we assume $|q| > 1/2b$, n is a positive integer). Then

$$\beta^{n-1} < 2b\,|q| \leq \beta^n$$

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and so

$$\left|\theta - \frac{p}{q}\right| > \frac{1}{2\,|q\,q_n|} > \frac{1}{2\,|q|\,a\,\alpha^n} = \frac{1}{2\,|q|\,a\,\beta^{\lambda n}} < \frac{1}{2\,|q|\,a\,(2|q|\,b\,\beta)^{\lambda}}.$$

If instead we have $p/q = p_n/q_n$ for our desired choice of n, we argue similarly, only with n replaced by $n+1$ (whereby the fact that our approximations are distinct guarantees that $p/q \neq p_{n+1}/q_{n+1}$). The slightly weaker constant in (1) results from this case.

An inequality of the shape

$$\left|\theta - \frac{p}{q}\right| > |q|^{-\kappa},$$

valid for suitably large integers p and q is termed an *irrationality measure*. For real transcendental θ, any such measure is in some sense nontrivial. For algebraic θ, say of degree n, however, Liouville's theorem provides a "trivial" lower bound of n for κ.

In this note, we intend to provide a perhaps oversimplified account of one technique for generating, for certain irrational θ, sequences p_n/q_n with the properties described above. In particular, we will discuss applications of the so-called *hypergeometric method*. An early example of use of this approach for problems in Diophantine approximation was was work of Thue (though the connection to hypergeometric functions is by no means apparent from a cursory read of [9]). More recently, it has been employed by Siegel [8], Baker [2] and many others (see [4] for a more extensive bibliography).

In the rest of this paper, we will show how to apply this method in two cases. The first is for $\theta = \zeta(2)$ and the second for certain algebraic θ of degree 4 which arise in a classical Diophantine problem of Ljunggren [7]. Neither of these examples is new, but in the second case, it is instructive to provide a proof that fits into a more general framework. It is this approach that has recently led Akhtari [1] to prove, via the techniques illustrated here, a conjecture of Walsh (motivated by [7]), to the effect that Diophantine equations of the shape

$$ax^4 - by^2 = 1$$

have, for fixed positive integers a and b, at most two solutions in positive integers x, y.

ALMOST AN EXAMPLE

The following argument is due to Beukers [5], inspired by Apery's proof of the irrationality of $\zeta(3)$.

We will sketch a proof of the irrationality of $\zeta(2) = \pi^2/6$. This is by no means the original proof, but it is rather instructive. To begin, we note the identity

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{r^2} + \int_0^1 \int_0^1 \frac{(xy)^r}{1-xy} dx\, dy.$$

To see this, just expand $1/(1-xy)$ as a geometric series, substitute and carry out the integrations. If, instead, we consider a similar integral, with distinct powers of x and y

in the numerator of the integrand, we find that, assuming, say, $r > s$ are nonnegative integers,

$$\int_0^1 \int_0^1 \frac{x^r y^s}{1-xy} dx\, dy = \sum_{k=0}^{\infty} \frac{1}{(k+r+1)(k+s+1)},$$

and so, via partial fractions and telescoping,

$$\int_0^1 \int_0^1 \frac{x^r y^s}{1-xy} dx\, dy = \frac{1}{r-s}\left(\frac{1}{s+1} + \cdots + \frac{1}{r}\right).$$

Defining

$$L(r) = \operatorname{lcm}(1,2,\ldots,r),$$

it follows that

$$L(r)^2 \int_0^1 \int_0^1 \frac{x^r y^s}{1-xy} dx\, dy$$

is thus, for each $r > s$, a positive integer.

Combining these facts, we find that, for any polynomial $P(x,y)$ with integer coefficients,

$$\int_0^1 \int_0^1 \frac{P(x,y)}{1-xy} dx\, dy = A\zeta(2) - B,$$

for *rational* A and B. We will, through careful choice of a family of such $P(x,y)$, construct our sequences of integers p_n and q_n such that p_n/q_n is a suitably good approximation to $\zeta(2)$.

Indeed, let us take

$$P(x,y) = (1-y)^n P_n(x),$$

where

$$P_n(x) = \frac{1}{n!}\left(\frac{d}{dx}\right)^n (x^n(1-x)^n).$$

Then, as noted before, we have

$$L(n)^2 \int_0^1 \int_0^1 \frac{(1-y)^n P_n(x)}{1-xy} dx\, dy = q_n \zeta(2) - p_n, \tag{2}$$

with p_n, q_n rational integers. One of the main reasons for this choice is that the integrand of the left-hand-side of (2) is now extremely small, while the coefficients of the numerator of the integrand do not grow too quickly. In fact, an n-fold integration by parts gives us the identity

$$\int_0^1 \int_0^1 \frac{(1-y)^n P_n(x)}{1-xy} dx\, dy = (-1)^n \int_0^1 \int_0^1 \frac{y^n(1-y)^n x^n(1-x)^n}{(1-xy)^{n+1}} dx\, dy$$

and since

$$\frac{y(1-y)x(1-x)}{1-xy} \le \left(\frac{\sqrt{5}-1}{2}\right)^5, \quad \text{for } 0 \le x,y \le 1,$$

we thus have

$$0<|q_n\zeta(2)-p_n|\le L(n)^2\left(\frac{\sqrt{5}-1}{2}\right)^{5n}\int_0^1\int_0^1\frac{dxdy}{1-xy}.$$

Since, $\log L(n)\sim n$ (via the Prime Number Theorem), and since

$$e^2\left(\frac{\sqrt{5}-1}{2}\right)^5<2/3,$$

it follows that $\zeta(2)$ is irrational.

If we work a little more carefully, we can estimate the growth of p_n and q_n and get an irrationality measure for $\zeta(2)$ (and hence for π) from this argument. What the reader should take from this "example" is the notion of constructing our approximating sequences p_n/q_n via specialization of rational functions. This idea will be worked out in greater detail in what follows.

PADÉ APPROXIMANTS TO $(1-z)^{1/m}$

Given a formal power series $f(z)$ and positive integers r and s, it is an exercise in linear algebra to deduce, for fixed integer n, the existence of nonzero polynomials $P_{r,s}(z)$ and $Q_{r,s}(z)$ with rational integer coefficients and degrees r and s, respectively, such that

$$P_{r,s}(z)-f(z)Q_{r,s}(z)=z^{r+s+1}E_{r,s}(z)$$

where $E_{r,s}(z)$ is a power series in z (let's not worry too much about convergence!). In certain situations, these *Padé approximants* (which are unique up to scaling) can be written down in explicit fashion. Such is the case for $f(z)=(1-z)^{1/m}$. Indeed, if we define, taking $r=s=n$ for simplicity,

$$P_n(z)=\sum_{k=0}^{n}\binom{n+1/m}{k}\binom{2n-k}{n}(-z)^k$$

and

$$Q_n(z)=\sum_{k=0}^{n}\binom{n-1/m}{k}\binom{2n-k}{kn}(-z)^k,$$

then there exists a power series $E_n(z)$ such that for all complex z with $|z|<1$,

$$P_n(z)-(1-z)^{1/m}Q_n(z)=z^{2n+1}E_n(z). \qquad (3)$$

How could we go about discovering these polynomials for ourselves? Let us write

$$I_n(z)=\frac{1}{2\pi i}\int_\gamma\frac{(1-wz)^{n+1/m}}{(w(w-1))^{n+1}}dw,$$

where γ is a closed counter-clockwise contour enclosing 0 and 1. Here, by $(1+t)^{1/m}$ for t a complex number with $|t| < 1$, we mean

$$(1+t)^{1/m} = \sum_{k=0}^{\infty} \binom{1/m}{k} t^k.$$

In fact, expanding the binomial series, we may write

$$I_n(z) = \sum_{h=0}^{\infty} \binom{n+1/m}{h} (-z)^h J_h,$$

where

$$J_h = \frac{1}{2\pi i} \int_{\gamma} \frac{w^{h-n-1}}{(w-1)^{n+1}} dw.$$

As is well-known, the integral of a rational function $P(w)/Q(w)$ over a closed contour enclosing its poles vanishes, provided the degree of the polynomial Q exceeds that of P by at least 2. We thus have $J_h = 0$ for $0 \leq h \leq 2n$. To find the shape of the coefficients of $P_n(z)$ and $Q_n(z)$ involves a residue calculation.

For later use, it is worth noting the following results. Proofs of these may be found in a number of places in the literature, including [3].

LEMMA 1 : Let n be a positive integer and suppose that z is a complex number with $|1-z| \leq 1$. Then

(i) We have

$$|P_n(z)| < 4^n, \; |Q_n(z)| < 4^n \text{ and } |E_n(z)| < 4^{-n}(1-|z|)^{-\frac{1}{2}(2n+1)}.$$

(ii) For all complex numbers $z \neq 0$, we have

$$P_n(z)Q_{n+1}(z) \neq P_{n+1}(z)Q_n(z).$$

(iii) If we define

$$\sigma_{k,m} = \prod_{p|m} p^{[k/(p-1)]},$$

then

$$\sigma_{k,m} m^k \binom{n+1/m}{k}$$

is an integer.

(iv) If we define $G_{n,m}$ to be the largest positive integer such that

$$\frac{\sigma_{n,m} m^n P_n(z)}{G_{n,m}} \text{ and } \frac{\sigma_{n,m} m^n Q_n(z)}{G_{n,m}}$$

are both polynomials with integer coefficients, then

$$G_{n,3} > \frac{1}{42} 2^n \text{ and } G_{n,4} > (3/2)^n,$$

for all positive integers n.

WHAT'S IN A NAME?

A *hypergeometric function* is a power series of the shape

$$F(\alpha,\beta,\gamma,z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)n!} z^n.$$

Here z is a complex variable and α, β and γ are complex constants. If α or β is a non-positive integer and m is the smallest integer such that

$$\alpha(\alpha+1)\cdots(\alpha+m)\beta(\beta+1)\cdots(\beta+m) = 0,$$

then $F(\alpha,\beta,\gamma,z)$ is a polynomial in z of degree m.

It is worth noting that the hypergeometric function $F(\alpha,\beta,\gamma,z)$ satisfies the differential equation

$$z(1-z)\frac{d^2F}{dz^2} + (\gamma - (1+\alpha+\beta)z)\frac{dF}{dz} - \alpha\beta F = 0. \qquad (4)$$

Note that, in terms of hypergeometric functions,

$$P_n(z) = \binom{2n}{n} F(-1/m-n, -n, -2n, z)$$

and

$$Q_n(z) = \binom{2n}{n} F(1/m-n, -n, -2n, z),$$

Another way to see (3), then, is to note that the functions

$$P_n(z),\ (1-z)^{1/m}Q_n(z) \ \text{ and } \ z^{2n+1}F(n+1, n+(m-1)/m, 2n+2, z)$$

each satisfy (4) with $\alpha = -1/m-n$, $\beta = -n$, $\gamma = -2n$, and hence, it is not too difficult to show, are linearly dependent over the rationals. Finding the dependency is then a reasonably easy exercise.

Most of the functions for which we are able to explicitly determine families of Padé approximants are special cases of the hypergeometric function.

A SECOND, HARDER EXAMPLE

We will, in what follows, prove the following

THEOREM 1 (Ljunggren, 1942) : If x and y are positive integers satisfying

$$x^2 + 1 = 2y^4,$$

then $(x,y) = (1,1)$ or $(x,y) = (239,13)$.

The original proof of this theorem (in [7]) utilized what might nowadays be considered a version of Chabauty's method. A proof of Theorem 1 via techniques akin to what we have been discussing was given by Chen [6] (though it's rather difficult to detect the connection from [6]). Let us begin by supposing that we have positive integers x and y for which

$$x^2 - 2y^4 = -1. \tag{5}$$

It follows that we can write

$$x + i = i^{\delta}(1+i)(a+ib)^4,$$

where we may suppose, without loss of generality, that a and b are positive integers and that $0 \leq \delta \leq 3$. Equating imaginary parts, arguing modulo 4, and renaming a and b if necessary, we may write

$$P(x,y) = a^4 + 4a^3b - 6a^2b^2 - 4ab^3 + b^4 = 1. \tag{6}$$

We note the solutions

$$(a,b) = (1,1) \text{ and } (3,2),$$

corresponding to the two known solutions $(x,y) = (1,1)$ and $(x,y) = (239,13)$, respectively, and suppose that (a,b) is distinct from these; our aim is to derive a contradiction. We note for future use that, after a short calculation, we may suppose $\min\{a,b\} > 10$.

To proceed, let us define $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$ to be linear functions of complex x and y, satisfying

$$\xi^4 = 4(i+1)(x-iy)^4 \quad \text{and} \quad \eta^4 = 4(i-1)(x+iy)^4.$$

We call (ξ,η), a pair of *resolvent forms*. Note that

$$P(x,y) = \frac{1}{8}(\xi^4 - \eta^4)$$

and if (ξ,η) is a pair of resolvent forms then there are precisely three others with distinct ratios, say $(-\xi,\eta)$, $(i\xi,\eta)$ and $(-i\xi,\eta)$.

Next, let us note that the roots of $P(x,1) = 0$ are given by $\beta_1 < \beta_2 < \beta_3 < \beta_4$, where

$$\beta_1 = -5.0273\ldots, \beta_2 = -0.6681\ldots, \beta_3 = 0.1989\ldots \text{and } \beta_4 = 1.4966\ldots$$

(so that $\beta_1\beta_3 = \beta_2\beta_4 = -1$). Notice further, since

$$P(\beta_i,1) = \frac{1}{8}(\xi^4(\beta_i,1) - \eta^4(\beta_i,1)) = 0,$$

that $\omega_i = \frac{\xi(\beta_i,1)}{\eta(\beta_i,1)}$ is a fourth root of unity, for each $1 \leq i \leq 4$. That $\omega_i \neq \omega_j$ for $i \neq j$ is an immediate consequence of the identity

$$\left| \frac{\xi(\beta_i,1)}{\eta(\beta_i,1)} - \frac{\xi(\beta_j,1)}{\eta(\beta_j,1)} \right| = \frac{2\left|(\beta_j - \beta_i)\right|}{\left|(\beta_i - i)(\beta_j - i)\right|}.$$

Fixing $i \in \{1,2,3,4\}$, a pair of resolvent forms ξ and η, and setting

$$z = z(x,y) = 1 - \left(\frac{\eta(x,y)}{\xi(x,y)}\right)^4,$$

we will say that the integer pair (x,y) is *related* to ω_i if

$$\left|\omega_i - \frac{\eta(x,y)}{\xi(x,y)}\right| < \frac{\pi}{12}|z(x,y)|.$$

It turns out that each nontrivial solution (x,y) to (6) is related to a fourth root of unity :

LEMMA 2 : Suppose that (x,y) is a positive integral solution to equation (6), with

$$\left|\omega_i - \frac{\eta(x,y)}{\xi(x,y)}\right| = \min_{0\leq k\leq 3}\left|e^{k\pi i/2} - \frac{\eta(x,y)}{\xi(x,y)}\right|.$$

Then

$$\left|\omega_i - \frac{\eta(x,y)}{\xi(x,y)}\right| < \frac{\pi}{12}|z(x,y)|. \tag{7}$$

To prove this, we begin by noting that

$$|z| = \left|\frac{\xi^4 - \eta^4}{\xi^4}\right| = \frac{8\,|P(x,y)|}{|\xi^4|} = \frac{8}{|\xi^4|},$$

and, from $xy \neq 0$,

$$|\xi^4(x,y)| = 4\sqrt{2}\,(x^2+y^2)^2 \geq 16\sqrt{2},$$

whereby $|z| < 1$. Since $\eta = -\overline{\xi}$, it follows that

$$\left|\frac{\eta}{\xi}\right| = 1, \;\; |1-z| = 1.$$

Defining $4\theta = \arg\left(\frac{\eta(x,y)^4}{\xi(x,y)^4}\right)$, we have

$$\sqrt{2-2\cos(4\theta)} = |z| < 1,$$

and so $|\theta| < \frac{\pi}{12}$. Since

$$\left|\omega_i - \frac{\eta(x,y)}{\xi(x,y)}\right| \leq |\theta|,$$

it follows that

$$\left|\omega_i - \frac{\eta(x,y)}{\xi(x,y)}\right| \leq \frac{1}{4}\frac{|4\theta|}{\sqrt{2-2\cos(4\theta)}}\left|1 - \frac{\eta(x,y)^4}{\xi(x,y)^4}\right|.$$

From the fact that

$$\frac{|4\theta|}{\sqrt{2-2\cos(4\theta)}} < \frac{\pi}{3}, \text{ whenever } 0 < |\theta| < \frac{\pi}{12},$$

we obtain inequality (7), as desired. This completes the proof of Lemma 2.

This lemma shows that each integer solution (x,y) to our Thue equation is related to precisely one fourth root of unity. We claim that $(x,y)=(3,2)$ is related to ω_4, while the pair $(x,y)=(1,5)$ (which is not actually a solution to (6)) is related to ω_3. To see this, notice that if we fix i, then

$$\left|\omega_i - \frac{\xi(x,y)}{\eta(x,y)}\right| = \left|\frac{\xi(\beta_i,1)}{\eta(\beta_i,1)} - \frac{\xi(\frac{x}{y},1)}{\eta(\frac{x}{y},1)}\right| = \frac{2\left|\beta_i - \frac{x}{y}\right|}{(\beta_i^2+1)((\frac{x}{y})^2+1)}, \tag{8}$$

whereby a short calculation verifies our claim. It is also easy to see that our putative larger positive solution (a,b) to (6) is related to either ω_3 or ω_4. Indeed, combining (7) and (8), we have that (a,b) is related to ω_i, where

$$\frac{2\left|\beta_i - \frac{a}{b}\right|}{(\beta_i^2+1)((\frac{a}{b})^2+1)} < \frac{\pi}{6\sqrt{2}(a^2+b^2)^2}.$$

It follows that

$$\left|\beta_i - \frac{a}{b}\right| < \frac{\pi(\beta_i^2+1)}{12\sqrt{2}b^2(a^2+b^2)}. \tag{9}$$

Since $\max\{\beta_1,\beta_2\} < -2/3$, the assumption that $\min\{a,b\} > 10$, thus implies that $i \in \{3,4\}$.

SOME ALGEBRAIC NUMBERS

To complete our proof, we will show that in fact $(x,y)=(1,5)$ and $(x,y)=(3,2)$ are the only integer pairs related to ω_3 and ω_4, respectively. We fix $(x,y,\omega)=(1,5,\omega_3)$ or $(3,2,\omega_4)$ and suppose, as the last section indicates that we may, that (a,b) is related to ω. For shorthand, let us write

$$\xi_1 = \xi(x,y), \ \ \eta_1 = \eta(x,y), \ \ \xi_2 = \xi(a,b), \ \ \eta_2 = \eta(a,b)$$

and set $z_1 = 1 - \eta_1^4/\xi_1^4$.

Combining our polynomials and resolvent forms of the previous sections, we define complex sequences Σ_n by

$$\Sigma_n = \frac{\eta_2}{\xi_2}P_n(z_1) - \frac{\eta_1}{\xi_1}Q_n(z_1).$$

The point in considering these sequences is as follows. Firstly, they are, in some sense, "close" to being Gaussian integers. In particular, at least provided $\Sigma_n \neq 0$, this enables us to give a nice, explicit lower bound upon $|\Sigma_n|$. On the other hand, from (3), we may write, for a well-chosen branch of $(z_1^4)^{1/4}$,

$$\Sigma_n = \left(\frac{\eta_2}{\xi_2} - \omega\right) P_n(z_1) + \omega z_1^{2n+1} E_n(z_1),$$

and so

$$|\Sigma_n| \leq \left|\omega - \frac{\eta_2}{\xi_2}\right| |P_n(z_1)| + |z_1|^{2n+1} |E_n(z_1)|. \tag{10}$$

Since both $\omega - \frac{\eta_2}{\xi_2}$ and z_1 are “small” in modulus, we hope to be able to derive an upper bound for $|\Sigma_n|$ that, in conjunction with our lower bound, yields useful information.

We will begin by finding our explicit lower bound. Since each of P_n and Q_n are polynomials of degree n with the property that

$$\frac{P_n(8z)}{G_{n,4}} \text{ and } \frac{Q_n(8z)}{G_{n,4}}$$

are polynomials with integer coefficients and since

$$(-2)^{-1/4}\xi_1\eta_2,\ (-2)^{-1/4}\xi_1\eta_2,\ \xi_1^4 \text{ and } \eta_1^4$$

are Gaussian integers, with

$$\xi_1^4 - \eta_1^4 = 8P(x_1, y_1),$$

it follows that

$$\Lambda_n = (-2)^{-1/4} G_{n,4}^{-1} \xi_1^{4n+1} \xi_2 \Sigma_n$$

is also a Gaussian integer. If $\Lambda_n \neq 0$ (equivalently, $\Sigma_n \neq 0$), this provides a lower bound upon $|\Lambda_n|$, namely $|\Lambda_n| \geq 1$. For a fixed choice of n, we necessarily have that at least one of Σ_n or Σ_{n+1} is nonzero; otherwise, it is easy to show that

$$P_n(z_1)Q_{n+1}(z_1) - P_{n+1}(z_1)Q_n(z_1) = 0,$$

contrary to Lemma 1.

If $\Sigma_n \neq 0$, then

$$2^{1/4} \leq G_{n,4}^{-1} |\xi_1|^{4n+1} |\xi_2| |\Sigma_n|$$

and hence, from (10),

$$2^{1/4} \leq \Gamma_1 + \Gamma_2, \tag{11}$$

where

$$\Gamma_1 = G_{n,4}^{-1} |\xi_1|^{4n+1} |\xi_2| \left|\omega - \frac{\eta_2}{\xi_2}\right| |P_n(z_1)|$$

and

$$\Gamma_2 = G_{n,4}^{-1} |\xi_1|^{4n+1} |\xi_2|\ |z_1|^{2n+1} |E_n(z_1)|.$$

In either case $(x,y) = (3,2)$ or $(1,5)$,

$$|z_1| = \frac{\sqrt{2}}{169} \text{ and either } |\xi_1|^4 = 26^2\sqrt{2} \text{ or } 52^2\sqrt{2},$$

respectively. We thus have

$$\Gamma_1 < \frac{4\pi}{3}\sqrt{13}\cdot 2^{1/8}\left(2^{15/2}3^{-1}13^2\right)^n |\xi_2|^{-3} < 17\cdot 21.7^{3n}|\xi_2|^{-3}.$$

Similarly,

$$\Gamma_2 < 0.07\cdot 22.2^{-n}|\xi_2|.$$

Inequality (11) thus implies that

$$1 < 15\cdot 21.7^{3n}|\xi_2|^{-3} + 0.06\cdot 22.2^{-n}|\xi_2| \tag{12}$$

holds for each positive integer n for which $\Sigma_n \neq 0$. For large enough n, if $|\xi_2|$ is roughly of size θ^n, for $\theta \in (21.7, 22.2)$, this inequality leads to the desired contradiction. To be precise, let us now choose a positive integer k such that

$$22^k < |\xi_2| \leq 22^{k+1}.$$

If $\Sigma_k \neq 0$, we take $k = n$ in (12) to conclude that

$$1 < 15\cdot 0.960^k + 1.32\cdot 0.991^k$$

and hence that $k \leq 88$. Similarly, if $\Sigma_k = 0$ then necessarily $\Sigma_{k+1} \neq 0$ and hence (12) with $n = k+1$ implies that $k \leq 292$. It follows that

$$2^{5/8}\sqrt{a^2+b^2} = |\xi_2| \leq 22^{293}$$

and so

$$a^2 + b^2 < e^{1811},$$

whereby $\max\{a,b\} < 10^{394}$.

FINISHING TOUCHES

We will finish our proof of Theorem 1 by showing that (6) has no solutions in integers a and b with

$$10 < b < 10^{394}.$$

As is well-known from the theory of continued fractions, given a real number β, if p and q are integers ($q > 0$) for which

$$\left|\beta - \frac{p}{q}\right| < \frac{1}{2q^2},$$

then necessarily p/q is a convergent on the infinite simple continued fraction to β. Moreover, if we have such a convergent p_m/q_m to β, with corresponding partial quotient a_{m+1}, then

$$\left|\beta - \frac{p_m}{q_m}\right| > \frac{1}{(a_{m+1}+2)\,q_m^2}. \tag{13}$$

In our situation, inequality (9) thus implies that, for either $\beta = \beta_3$ or β_4, a/b is a convergent to β, say $a/b = p_m/q_m$. Since a and b are positive and coprime, $a = p_m$ and $b = q_m$. Combining (9) and (13), we find that

$$a_{m+1} > \frac{12\sqrt{2}}{\pi(\beta_i^2+1)} q_m(p_m^2+q_m^2) - 2.$$

Since $\beta_i < 3/2$ and $\min\{a,b\} > 10$, it follows that

$$a_{m+1} \geq 10443.$$

On the other hand, a short calculation with any standard computational package shows that, for $\beta = \beta_3$, $q_m < 10^{394}$ only for $m \leq 798$ and that the largest corresponding a_{m+1} in this range is $a_{102} = 302$. For $\beta = \beta_4$, we have $m \leq 764$ with maximal $a_{90} = 604$. This contradiction completes our proof of Theorem 1.

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